

Computer Graphics: CO 303
Lecture 6, 7

Draconifors

Rolls

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Transformation as Matrices

$$\begin{array}{ccc} s_x & 0 & x \\ 0 & s_y & y \end{array} = \begin{array}{l} s_x \cdot x \\ s_y \cdot y \end{array}$$

$$\begin{array}{ccc} \cos \theta & -\sin \theta & x \\ \sin \theta & \cos \theta & y \end{array} = \begin{array}{l} x \cos \theta - y \sin \theta \\ x \sin \theta + y \cos \theta \end{array}$$

$$\begin{array}{ccc} t_x & + & x \\ t_y & + & y \end{array} = \begin{array}{l} x + t_x \\ y + t_y \end{array}$$

Transformation as Matrices

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$$\begin{array}{ccc} t_x & + & x \\ t_y & + & y \end{array} = \begin{array}{l} x + t_x \\ y + t_y \end{array}$$

Only translation cannot be expressed as matrix-vector multiplications.

Homogenous Coordinates

- In order to represent translation as a matrix multiplication operation we use 3 x 3 matrices and pad our points to become 3 x 1 matrices.

$$Trans(d_x, d_y) = \begin{bmatrix} 1 & 0 & d_x \\ 0 & 1 & d_y \\ 0 & 0 & 1 \end{bmatrix} \quad Scale(s_x, s_y) = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$Rot(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad Shear(a_x) = \begin{bmatrix} 1 & a_x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Connection to 3D?
- $(x, y, 1)$ represents a 3D point on the plane $W = 1$.
A homogeneous point is a **line in 3D**, through the origin

Composite Transformations

- Suppose we wish to perform multiple transformations on a point:

$$P_2 = T_{3,1}P_1$$

$$P_3 = S_{2,2}P_2$$

$$P_4 = R_{30}P_3$$

- But we want something like this:

$$M = R_{30}S_{2,2}T_{3,1}$$

$$P_4 = MP_1$$

The first transformation you want to perform will be at the far right, just before the point.
[Matrix multiplication is associative, not commutative!]

Composite Transformations

- Suppose we wish to perform multiple transformations on a point:

$$P_2 = T_{3,1}P_1$$

$$P_3 = S_{2,2}P_2$$

$$P_4 = R_{30}P_3$$

- But we want something like this:
 - Concatenate basic transforms sequentially.

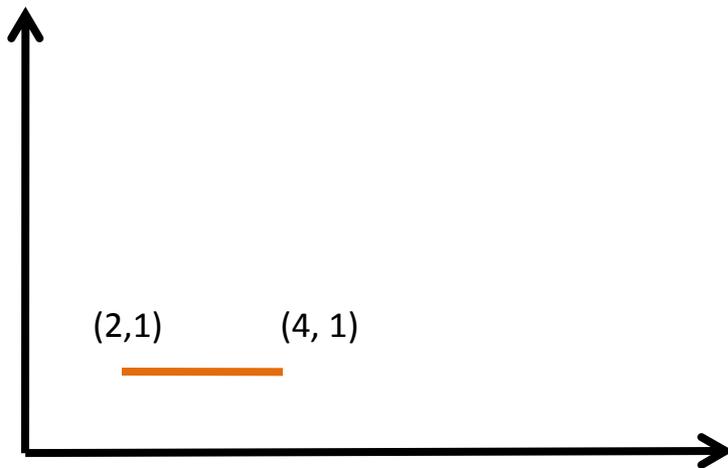
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$$P_4 = MP_1$$

The first transformation you want to perform will be at the far right, just before the point.
[Matrix multiplication is associative, not commutative!]

Example: Fixed Point Scaling

- Whenever scale transformation is performed, it also moves the object being scaled.
 - Example: Scale a line between $(2, 1)$ $(4, 1)$ to twice its length.



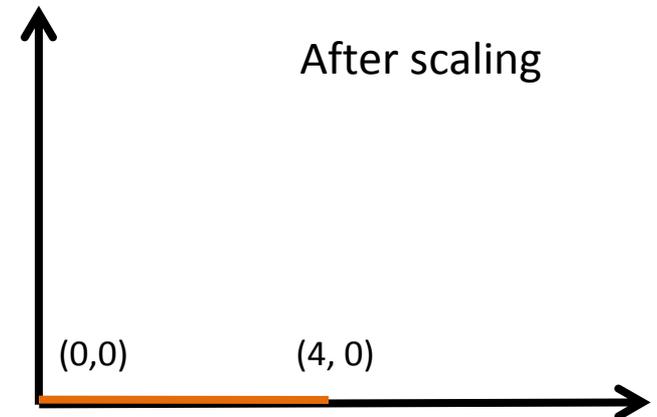
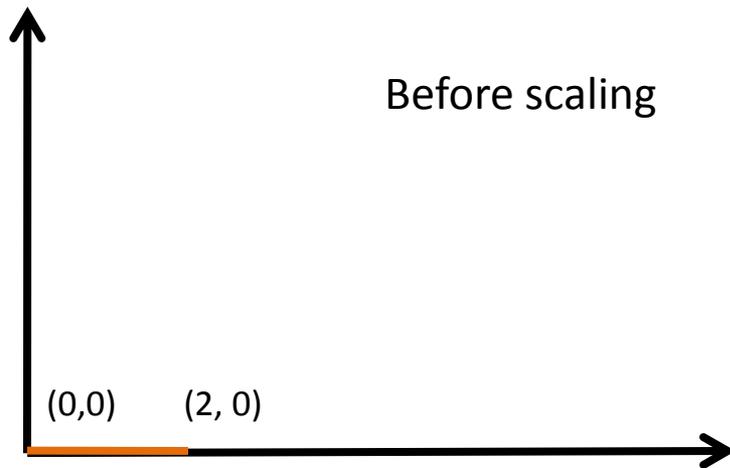
Original



After scaling

Example: Fixed Point Scaling

- If we scale a line between $(0,0)$ & $(2,0)$ to twice its length, the left-hand endpoint does not move.

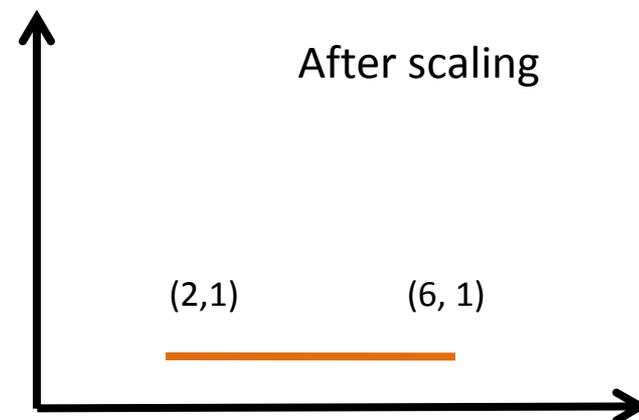
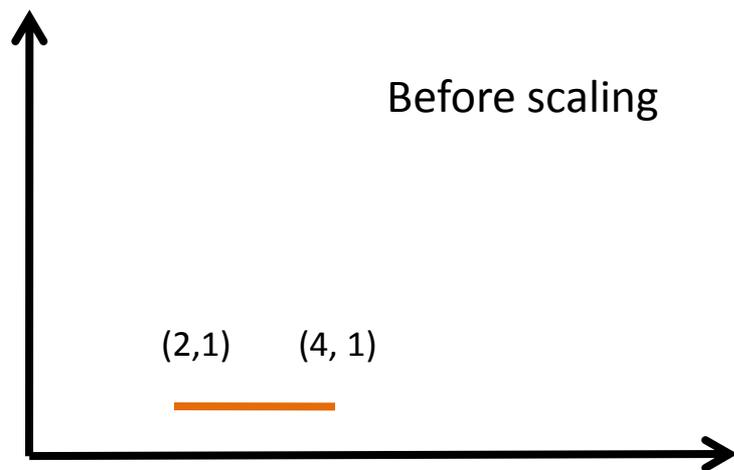


$(0,0)$ is known as a fixed point for the basic scaling transformation.

We can use composite transformations to create a scale transformation with different fixed points.

Example: Fixed Point Scaling

- Scale by 2 with fixed point = (2,1)
- Translate the point (2,1) to the origin
- Scale by 2
- Translate origin to point (2,1)



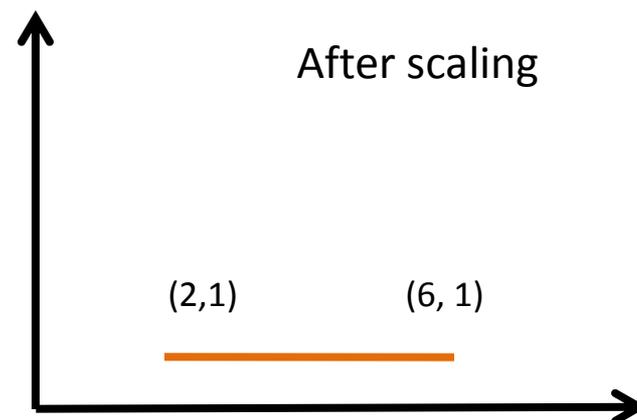
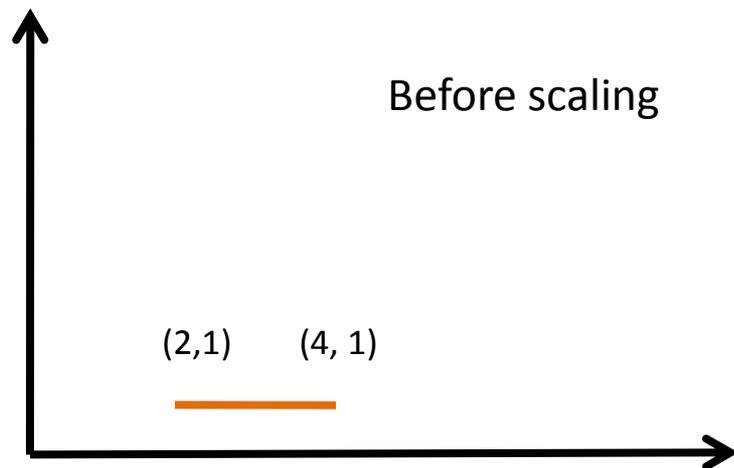
Example: Fixed Point Scaling

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$T_{2,1}$ $S_{2,1}$ $T_{-2,-1}$ C

$$\begin{bmatrix} 2 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 2 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \\ 1 \end{bmatrix}$$

C C



Compound Translation

- What happens when a point goes through

$T(d_{x1}, d_{y1})$ and then $T(d_{x2}, d_{y2})$?

- Combined translation: $T(d_{x1}+d_{x2}, d_{y1}+d_{y2})$

$$\begin{bmatrix} 1 & 0 & d_{x2} \\ 0 & 1 & d_{y2} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & d_{x1} \\ 0 & 1 & d_{y1} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & d_{x1} + d_{x2} \\ 0 & 1 & d_{y1} + d_{y2} \\ 0 & 0 & 1 \end{bmatrix}$$

Concatenation of transformations: **matrix multiplication**

Compound Rotation

- What happens when a point goes through $R(\theta)$ and then $R(\phi)$?

- Combined rotation: $R(\theta + \phi)$

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos(\theta + \phi) & -\sin(\theta + \phi) & 0 \\ \sin(\theta + \phi) & \cos(\theta + \phi) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Scaling or shearing (in one direction) is similar
- These concatenations are **commutative**

Commutativity

- Transformations that commute
 - Translate-Translate
 - Rotate-Rotate (around the same axis)
 - Scale-Scale
 - Uniform scale-Rotate
 - Shear in X (or Y) - Shear in X (or Y)

- *The first transformation you want to perform will be at the far right, just before the point.*

Inverse of Transformations

- Inverse of $T(d_x, d_y) = T(-d_x, -d_y)$

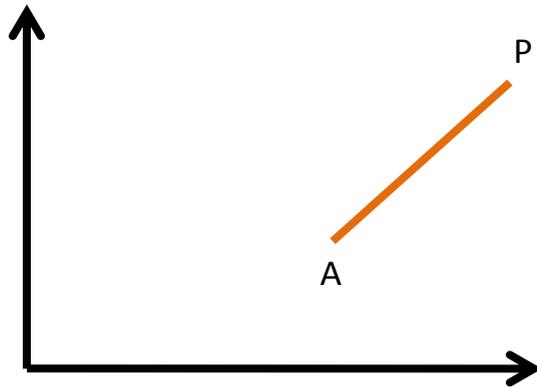
$$\begin{bmatrix} 1 & 0 & d_x \\ 0 & 1 & d_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -d_x \\ 0 & 1 & -d_y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T(-d_x, -d_y) = T(d_x, d_y)^{-1}$$

- $R(\theta)^{-1} = R(-\theta)$
- $S(s_x, s_y)^{-1} = S(1/s_x, 1/s_y)$
- $\text{Sh}(a_x)^{-1} = \text{Sh}(-a_x)$

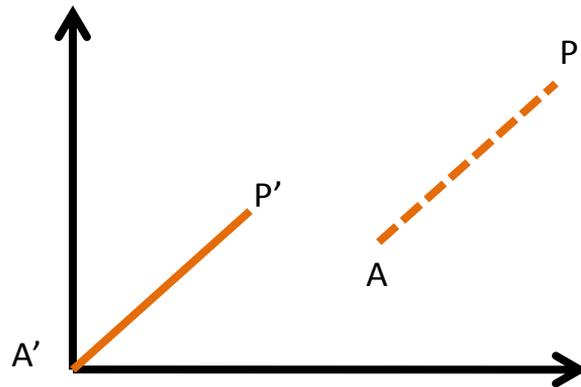
Example: Rotation about arbitrary axis

- Break it down into easier problems
 - Translate to the origin: $T(-x_1, -y_1)$



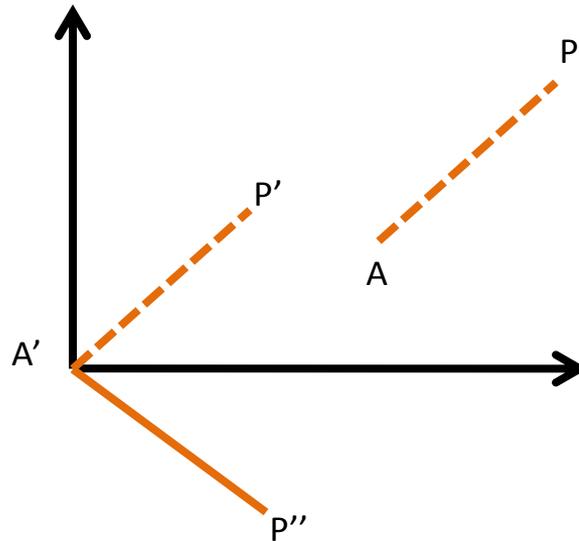
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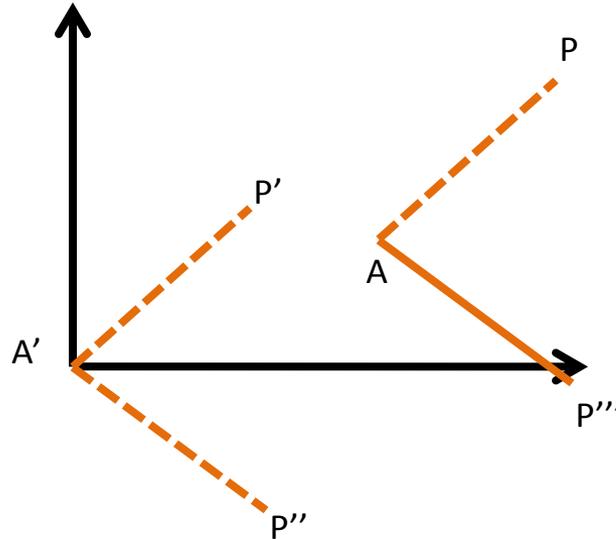
Example: Rotation about arbitrary axis

- Break it down into easier problems
 - Translate to the origin: $T(-x_1, -y_1)$
 - Rotate: $R(q)$



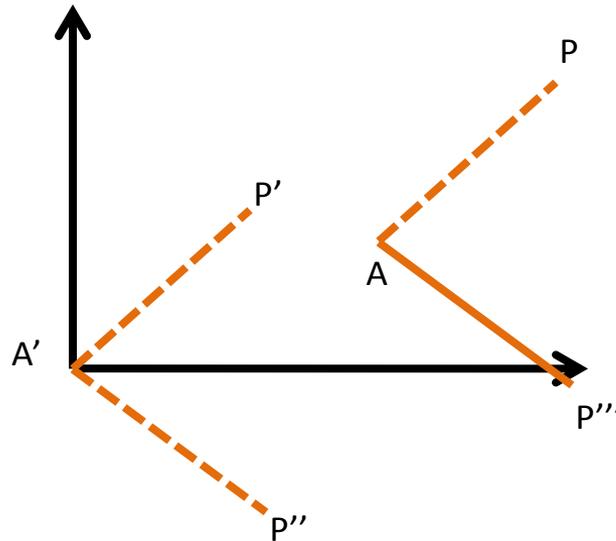
Example: Rotation about arbitrary axis

- Break it down into easier problems
 - Translate to the origin: $T(-x_1, -y_1)$
 - Rotate: $R(q)$
 - And translate back: $T(x_1, y_1)$



Example: Rotation about arbitrary axis

- Break it down into easier problems
 - Translate to the origin: $T(-x_1, -y_1)$
 - Rotate: $R(q)$
 - And translate back: $T(x_1, y_1)$



$$M = T(A_{x'}, A_{y'}) R(-90) T(-A_{x'}, -A_{y'})$$

Example: Rotation about arbitrary axis

- Rotation of θ Degrees About Point (x,y)

$$M = T(A_x, A_y) R(\theta) T(-A_x, -A_y)$$

$$C = \begin{matrix} \begin{bmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \\ T_{x,y} \end{matrix} \begin{matrix} \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ R_\theta \end{matrix} \begin{matrix} \begin{bmatrix} 1 & 0 & -x \\ 0 & 1 & -y \\ 0 & 0 & 1 \end{bmatrix} \\ T_{-x,-y} \end{matrix}$$

Example: Rotation about arbitrary axis

- Rotation of θ Degrees About Point (x,y)
 - Reflection about Y axis

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Reflection about X axis

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Affine Transformations in 3D

- General Form

$$\begin{pmatrix} Q_x \\ Q_y \\ Q_z \\ 1 \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} P_x \\ P_y \\ P_z \\ 1 \end{pmatrix}$$

Affine Transformations in 3D

- General Form

$$\begin{pmatrix} Q_x \\ Q_y \\ Q_z \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & T_x \\ 0 & 1 & 0 & T_y \\ 0 & 0 & 1 & T_z \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} P_x \\ P_y \\ P_z \\ 1 \end{pmatrix}$$

Affine Transformations in 3D

- Scaling around the origin

$$\begin{pmatrix} Q_x \\ Q_y \\ Q_z \\ 1 \end{pmatrix} = \begin{pmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} P_x \\ P_y \\ P_z \\ 1 \end{pmatrix}$$

Affine Transformations in 3D

- Shear around the origin
 - Along the X-axis

$$\begin{pmatrix} Q_x \\ Q_y \\ Q_z \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & a & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} P_x \\ P_y \\ P_z \\ 1 \end{pmatrix}$$

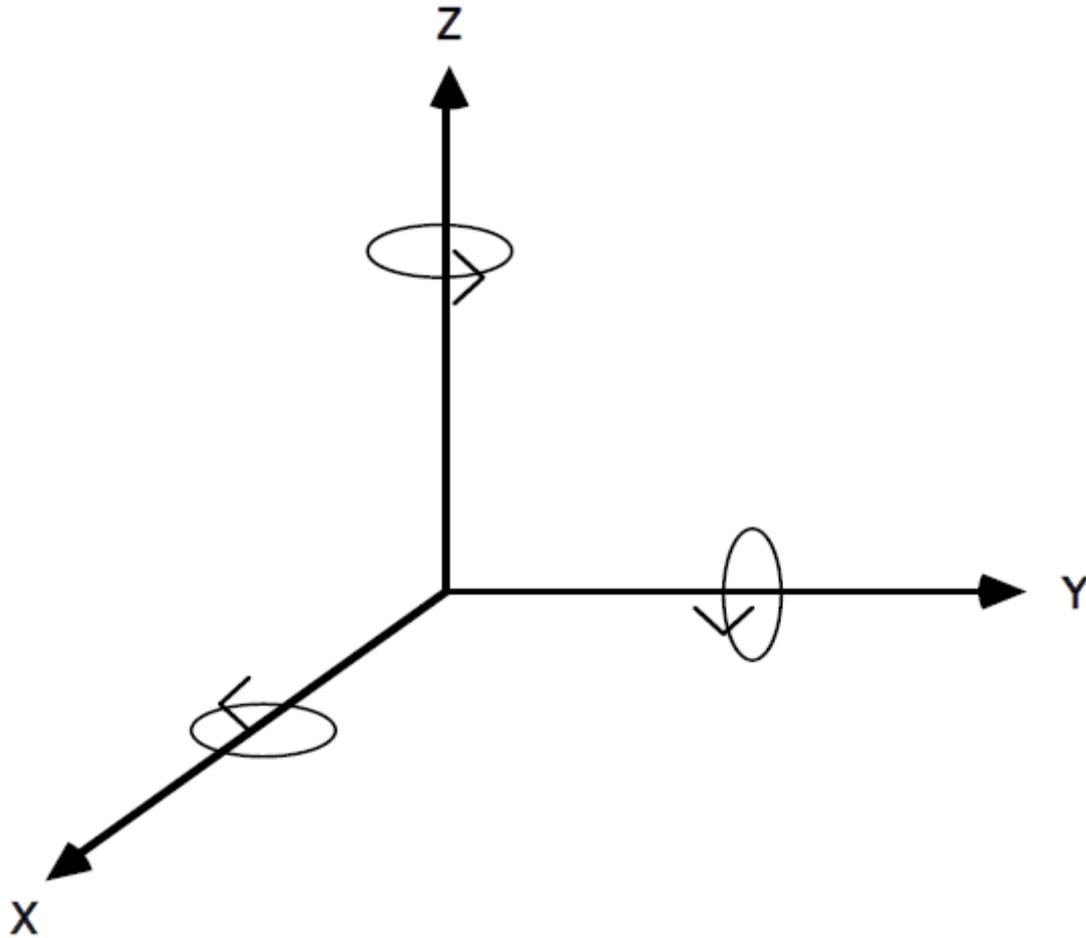
Reminder: 2D rotation (z-roll)

$$Q_x = \cos\theta P_x - \sin\theta P_y$$

$$Q_y = \sin\theta P_x + \cos\theta P_y$$

$$\begin{pmatrix} Q_x \\ Q_y \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} P_x \\ P_y \\ 1 \end{pmatrix}$$

3 axes to rotate around



Z roll

$$Q_x = \cos\theta P_x - \sin\theta P_y$$

$$Q_y = \sin\theta P_x + \cos\theta P_y$$

$$Q_z = P_z$$

$$R_z(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 & 0 \\ \sin(\theta) & \cos(\theta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

X roll

$$x \rightarrow y \rightarrow z \rightarrow x \rightarrow y$$

$$Q_y = \cos\theta P_y - \sin\theta P_z$$

$$Q_z = \sin\theta P_y + \cos\theta P_z$$

$$Q_x = P_x$$

$$R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) & 0 \\ 0 & \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Y roll

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \rightarrow \begin{bmatrix} x \\ y \\ z \\ x \\ y \end{bmatrix}$$

$$Q_z = \cos\theta P_z - \sin\theta P_x$$

$$Q_x = \sin\theta P_z + \cos\theta P_x$$

$$Q_y = P_y$$

$$R_y(\theta) = \begin{pmatrix} \cos(\theta) & 0 & \sin(\theta) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

THANKS!